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COMMENT

Non-Gaussian random walks

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Abstract. We present an explicit expression for the probability distribution for the position of a continuous-time random walker in an arbitrary number of dimensions when the interjump density has a long time tail, in contrast to earlier results which require numerical inversion of a Fourier integral. We replace this numerical procedure by one that relies on the method of steepest descents. Our results are applied to diffusion on a comb and on a percolation cluster generated on a Cayley tree at criticality and are confirmed numerically.

Forms for the probability distributions for diffusion on a fractal have been suggested recently by a number of investigators with somewhat differing results [1-5]. It is known that diffusion on a fractal is generally anomalous, but the detailed structure of the state density $p(\boldsymbol{r}, t)$ is still a matter of dispute. Two of us [5] have investigated diffusion on a comb structure having some of the important features associated with a finitely ramified fractal structure, namely the comb has a single backbone and a set of teeth that are the analogue of dead ends on the fractal. In our earlier paper [5] we showed that the mean square displacement in the absence of a field is asymptotically $\langle x^2\rangle \sim t^{1/2}$. However, we were in error in identifying the probability distribution for the displacement along the backbone as a Gaussian. In the present comment we derive the correct asymptotic form for $p(x, t)$ as well as its generalisations to higher dimensions, by taking advantage of the representation of the random walk as a continuous-time random walk (CTRW). The same analysis can be done in discrete time with no essential difference in results.

Tunaley [6] and Shlesinger *et al* [7] calculated forms of $p(x, t)$ (in 1D) in terms of the probability distribution for stable law forms for the density interjump times. The forms of the results require numerical calculations, which Tunaley carried out for a few particular cases by using the fast Fourier transform. We will complement his results by finding an expression in closed form for $p(\boldsymbol{r}, t)$ by evaluating the crucial inversion integral using the method of steepest descents. We then relate our results to the problem of diffusion on fractals.

Let us therefore consider a D-dimensional CTRW which for simplicity will be assumed to take place on a translationally invariant lattice. Let $p(j)$ be the single-step probability, i.e. $p(j)$ is the probability that the displacement of the random walker is equal to j in any given step. Let $\psi(t)$ be the probability density for the time between successive steps. In what follows we assume that $\psi(t)$ has a long tail in the sense that

$$\psi(t) \sim T^{\alpha}/t^{\alpha+1} \qquad 0<\alpha<1 \tag{1}$$

for $t \gg T$, where T is a constant. We further define the quantities

$$\lambda(\Theta) = \sum p(\boldsymbol{j}) \exp(\mathrm{i}\boldsymbol{j} \cdot \Theta) \tag{2}$$

$$\psi(s) = \int_0^\infty \exp(-st)\psi(t)\,\mathrm{d}t.$$

Let $\hat{p}(\boldsymbol{r}, s)$ denote the Laplace transform with respect to t of $p(r, t)$. Montroll and Weiss [8] have shown that

$$\hat{p}(\boldsymbol{r}, s) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{(2\pi)^D} \int_{-\pi}^{\pi} \cdots \int \frac{\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\theta})}{1-\hat{\psi}(s)\lambda(\boldsymbol{\theta})} \mathrm{d}^D\boldsymbol{\theta}. \tag{3}$$

In order to find the asymptotic form for $p(r, t)$ we study the behaviour of $\hat{p}(\boldsymbol{r}, s)$ for $s \to 0$. Since $r = |\boldsymbol{r}\cdot\boldsymbol{r}|^{1/2}$ will also be large when $t \to \infty$ we must also study the behaviour of the integrand in (3) in the limit $|\Theta|^2 \to 0$. When $s \to 0$ the form in (1) implies that

$$\psi(s) \sim 1 - (sT_0)^\alpha + \mathrm{O}(s^\alpha) \tag{4}$$

where T_0 is a constant with the dimensions of time. If we further assume that the lattice random walk is isotropic and the variance associated with each step is finite,

$$\sum_{j=-\infty}^{\infty} j^2 p(j) = \sigma^2 < \infty \tag{5}$$

then for Θ^2 small in an isotropic random walk we have the expansion

$$\lambda(\Theta) \sim 1 - (\sigma^2/2) \sum_{j=1}^{D} \Theta_j^2. \tag{6}$$

We may then approximate $\hat{p}(\boldsymbol{r}, s)$ by the integral representation

$$\hat{p}(\boldsymbol{r}, s) \sim \frac{T_0^\alpha s^{\alpha-1}}{(2\pi)^D} \int_{-\infty}^{\infty} \cdots \int \frac{\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\theta})}{(sT_0)^\alpha + \frac{1}{2}\sigma^2 \sum_{j=1}^{D} \theta_j^2} \mathrm{d}^D\boldsymbol{\theta}. \tag{7}$$

This integral can be evaluated by making use of the identity

$$u^{-1} = \int_0^\infty \exp(-u\xi)\,\mathrm{d}\xi \tag{8}$$

to factorise the denominator appearing in (7). In this way we find

$$\hat{p}(\boldsymbol{r}, s) \sim \frac{2^{(1/2)(1+D/2)}}{(2\pi\sigma^2)^{D/2}} \left(\frac{r}{\sigma}\right)^{1-D/2} \frac{T_0}{(sT_0)^{(\alpha/2)(1-D/2)}} K_{1-D/2}\left(\frac{2^{1/2}r}{\sigma}(sT_0)^{\alpha/2}\right) \tag{9}$$

where $K_{1-D/2}(x)$ is a Bessel function of the second order of the imaginary argument.

Let us consider the case $D = 1$ in the last equation which leads to the result

$$\hat{p}(x, s) \sim \frac{1}{\sqrt{2}} \left(\frac{T_0}{s}\right)^{1-\alpha/2} \frac{1}{\sigma} \exp\left(-\frac{2^{1/2}x}{\sigma}(sT_0)^{\alpha/2}\right) \tag{10}$$

evaluated along the Bromwich contour. Since our expression in (7) is only valid for large t we may use the method of steepest descents to derive an approximate expression for $p(x, t)$ from the Laplace transform inversion formula

$$p(x, t) \sim \frac{1}{2\pi\mathrm{i}} \int \frac{\hat{p}(x, s)}{s} \exp(st)\,\mathrm{d}s. \tag{11}$$

The equation for the root of the steepest descents equation is s_0, where, aside from trivial constants,

$$s_0=(x/t)^{2/(2-\alpha)} \tag{12}$$

which implies that $p(x, t)$ has the form

$$p(x, t)\sim t^{-\alpha/2}\exp[-(x^2/t^\alpha)^{1/(2-\alpha)}] \tag{13}$$

where the constants have again been ignored. In higher dimensions a distinction needs to be drawn between D odd and D even. When D is odd $K_{1-D/2}(u)$ can be written as a polynomial in u^{-1} multiplied by $\exp(-u)/u^{1/2}$ so that the equation for s_0 is the same as that for $D=1$ with x replaced by r. In this case one finds a slightly more complicated expression for $p(\boldsymbol{r}, t)$ than that given in (13) but the exponential term has the same form as that in (13). For example, when $D=3$ one finds (omitting inessential constants)

$$rp(r, t)\sim t^{-3\alpha/2}[r+(t/r)^{\alpha/(2-\alpha)}]\exp[-(r^2/t^\alpha)^{1/(2-\alpha)}]. \tag{14}$$

In even numbers of dimensions the equation for s_0 is considerably more difficult to solve for s_0 since the steepest descents equation is transcendental. However $p(\boldsymbol{r}, t)$ will contain the same exponential term as in (14) in the tails of the distribution.

If the diffusion exponent d_w is defined by $\langle x^2\rangle\sim t^{2/d_w}$ then the parameter d_w can be expressed in terms of α by $d_w=2/\alpha$ and the exponent in (13) can be expressed as $-(x/t^{1/d_w})^{d_w/(d_w-1)}$.

We have investigated several applications of these general results. For diffusion on the backbone of the infinite comb we have shown that $d_w=4$ or $\alpha=\frac{1}{2}$. Consequently the probability distribution for displacement along the backbone will have the form

$$p(x, t)\sim t^{-1/4}\exp(-x^{4/3}/t^{1/3}) \tag{15}$$

rather than the Gaussian form assumed in [5]. This expression for $p(x, t)$, has been confirmed for several values of t by numerical simulations using the exact enumeration method [9] (see figure 1). In the more general case of a comb with teeth of random length, the distribution of the length of a single tooth having the property

$$\varphi(L)\sim L^{-(1+\gamma)} \qquad 0<\gamma<1 \tag{16}$$

for large L then it was found [10] that $d_w=4/(1+\gamma)$ so that

$$p(x, t)\sim t^{-(1+\gamma)/4}\exp[-(x^4/t^{1+\gamma})^{1/(3-\gamma)}]. \tag{17}$$

The case of $\gamma=0$ corresponds to the comb with infinite teeth and $\gamma>1$ corresponds to a random comb whose teeth have a finite mean length. The limiting form for the distribution in this case is reached at $\gamma=1$, corresponding to a Gaussian form for $p(x, t)$

A second example to which our general results have been applied is that of a random walk on the infinite cluster generated on a Cayley tree at criticality. In this case the backbone of the cluster in the chemical distance metric [11] can be regarded as the backbone of a comb model, with random delays due to time spent in the dead ends. The distribution of cluster sizes has been shown to have the asymptotic form [12]

$$n(s)\sim s^{-3/2} \tag{18}$$

If we assume the transition rate for returning to the backbone $w(s)$ is inversely proportional to the cluster size then the distribution of such rates at low values of w varies as

$$h(w)=n(s)\,\mathrm{d}s/\mathrm{d}w\sim w^{-1/2} \tag{19}$$

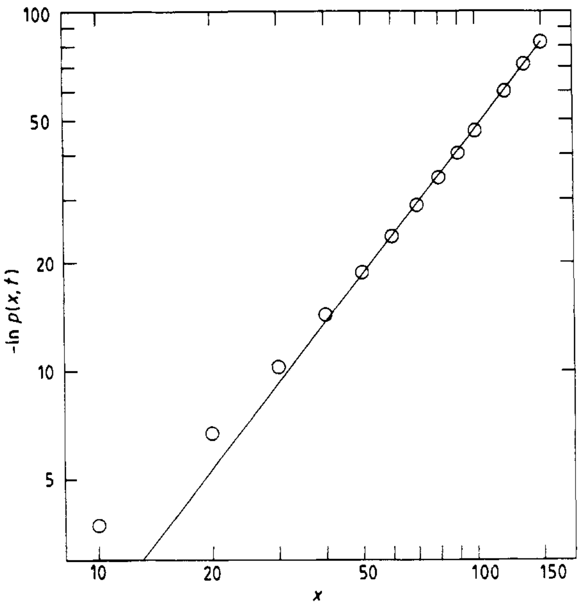

Figure 1. Plot of log(log $p(x, t)$) as a function of log x ($t = 400$) for diffusion on the infinite comb. The slope of this curve is 1.34 ± 0.02 as against the predicted value of $\frac{4}{3}$ (equation (15)).

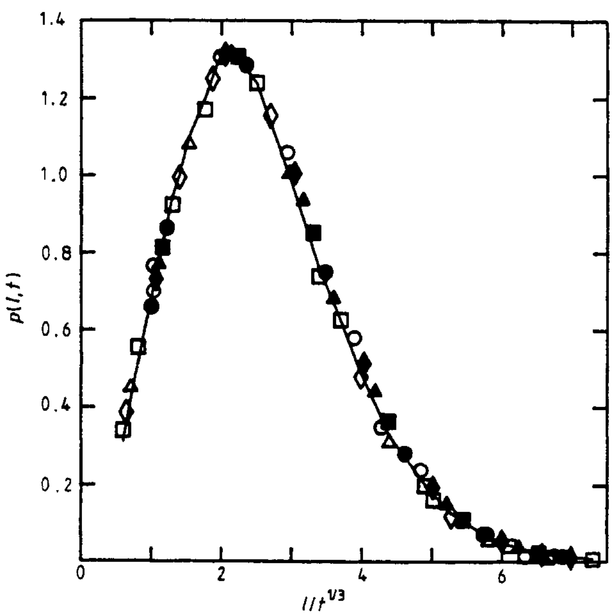

Figure 2. Plot of $p(l, t)$ as a function of $l/t^{1/d_w^l}$ for different values of l and t. The different symbols specify different numbers of steps: $t = 400$ (○), 800(◇), 1200 (+), 1600 (△), 2000 (□), 2400 (×), 2800 (▽), 3200 (■), 3600 (●), 4000 (▲). The range of the chemical distance l is $10 \leq l \leq 200$. The full curve represents the best fit to (20) with the parameters $d_l = 2.02 \pm 0.05$, $d_s = 1.35 \pm 0.05$, $d_w^l = 3.0 \pm 0.10$.

which corresponds to $\alpha = \frac{1}{2}$ in (1). In this case because of the self-similarity of the fractal can be shown [10] that one must use the Alexander *et al* result [13] $d_w = (1+\alpha)/\alpha$, rather than the CTRW result, $d_w = 2/\alpha$, that $d_w = 3$ for diffusion on the Cayley tree. Thus we expect that the probability density for the chemical distance of the random walker at time t is

$$p(l, t) \sim (l^{d_l - 1}/t^{d_s/2}) \exp[-(l/t^{1/d_w^l})^{d_w^l/(d_w^l - 1)}] \tag{20}$$

with $d_l = 2$, $d_s = \frac{4}{3}$ and $d_w^l = 3$. This equation has been tested numerically with results plotted in figure 2. The data strongly support the form of the equation given in (13) with the parameters just specified.

Finally, we mention two other problems that can be treated by the same techniques. The first is that of the biased random walk on the comb structure and the second is that of the random walk with steps whose average step length is infinite. The latter has been discussed by Schlesinger *et al* [7] and results can be derived for the probability density of displacement leading to different exponents than those in (13). However, the detailed calculations for the two problems require somewhat more complicated numerical calculations than those presented here.

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